PD Dr. Maksim Zhykhovich

Winter Term 2025/26
05.02.2025

Algebraic geometry 1

Exam

Family Name: ______________ First name: ______________

Student ID: ______________ Term: ______________

Degree course: Bachelor, PO ❑ 2011 ❑ 2015 ❑ 2021 Master, PO ❑ 2011 ❑ 2021

Lehramt Gymnasium: ❑ modularisiert ❑ nicht modularisiert

❑ Diplom ❑ Other: ______________

Major subject: ❑ Mathematik ❑ Wirtschaftsm. ❑ Inf. ❑ Phys. ❑ Stat. ❑ ______

Minor subject: ❑ Mathematik ❑ Wirtschaftsm. ❑ Inf. ❑ Phys. ❑ Stat. ❑ ______

Credit Points to be used for ❑ Hauptfach ❑ Nebenfach (Bachelor / Master)

Please switch off your mobile phone and do not place it on the table; place your identity and student ID cards on the table so that they are clearly visible.
Please check that you have received all **five problems**.
Please do not write with the colours red or green. Write **on every page** your **family name and your first name**.
Write your solutions on the page marked with the appropriate problem number. If you run out of space, use the empty pages at the end of the examination paper ensuring that each problem is clearly marked.
Please make sure to submit only one solution for each problem; cross out everything that should not be graded.

Good luck!

1	2	3	4	5	$\sum$
/9	/19	/7	/15	/14	/64

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Problem 1. **[4+5=9 Points]**

1) Let $X \subset \mathbb{A}^n$ be an affine algebraic set and let $y \in \mathbb{A}^n$ be a point, such that $y \notin X$.
Show that there exists a polynomial $F \in I(X)$, such that $F(y) \neq 0$.

2) A topological space X is *Hausdorff* if for any two distinct points $x, y \in X$, there exist open subsets $U, V \subset X$, such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.
Let $X \subset \mathbb{A}^n$ be an affine algebraic set endowed with the Zariski topology.
Show that if X is Hausdorff, then X is a finite union of points.
Hint: Consider the decomposition of X as a union of its irreducible components.

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Problem 2. **[1+2+3+2+6+2+3=19 Points]**

Let K be an algebraically closed field.
Let $L = \{(t, t, t) \mid t \in K\}$ be a subset in $\mathbb{A}^3 = \{(x, y, z) \mid x, y, z \in K\}$.

1) Show that $L = V(X - Y, Y - Z)$. In particular, L is an affine algebraic set.

2) Consider the homomorphism of K-algebras

$$\theta : K[X, Y, Z] \to K[T], X \mapsto T, Y \mapsto T, Z \mapsto T.$$

Show that $I(L) = \text{Ker } \theta$.

3) Show that $I(L) = (X - Y, Y - Z)$.

4) Show that L is isomorphic to $\mathbb{A}^1$ and find $\dim L$.

5) Denote by U the open subset $\mathbb{A}^3 \setminus L$ in $\mathbb{A}^3$.
Let f be a regular function on U (that is $f \in \mathcal{O}_{\mathbb{A}^3}(U) = A(U)$).
Show that there exists a polynomial $F \in K[X, Y, Z]$, such that $f(a) = F(a)$ for every $a \in U$ (that is $f = F_{|U}$).

6) Denote by $i : U \to \mathbb{A}^3$ the natural inclusion.
Show that $i^* : A(\mathbb{A}^3) \to A(U), g \mapsto g \circ i = g_{|U}$, is an isomorphism of K-algebras.

7) Show that U is not isomorphic to an affine algebraic set.
Hint: Assume that there is an isomorphism $\varphi : V \to U$, where V is an affine algebraic set. Consider the composition $i \circ \varphi : V \to U \to \mathbb{A}^3$.

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Problem 3. **[2+5 = 7 Points]**

Let $U_i := \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 \mid x_i \neq 0\}$, $i = 0, 1, 2$, be open subsets in $\mathbb{P}^2$ and set $U := U_0 \cap U_1 \cap U_2$.
1) Explain why the map

$$\varphi : U \to U,\ [x_0 : x_1 : x_2] \mapsto [x_1 x_2 : x_0 x_2 : x_0 x_1]\,,$$

is well-defined and is a morphism of quasi-projective algebraic sets.
2) Show that φ is an isomorphism.

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Problem 4. **[8+7=15 Points]**

1) Let $X \subset \mathbb{P}^3$ be a projective irreducible conic (that is $\dim X = 1$ and $\deg X = 2$).
Show that there exists a hyperplane $H \subset \mathbb{P}^3$ containing X.
Hint: Use Bézout's Theorem.

2) Find an example of a projective irreducible cubic $X \subset \mathbb{P}^3$ (that is $\dim X = 1$ and $\deg X = 3$), which is not contained in any hyperplane $H \subset \mathbb{P}^3$.
Hint: Let $v_d : \mathbb{P}^n \to \mathbb{P}^N$ be the Veronese embedding of degree d. You can use that $\deg v_d(\mathbb{P}^n) = d^n$ (this was proven in the exercises).

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Problem 5. **[6+8=14 Points]**

1) Let $F = X^2 + Y^2 - Z^2$ and $G = X^2 + Y^2 - Y + Z^2$ be two irreducible polynomials in $\mathbb{C}[X, Y, Z]$. Let $W \subset \mathbb{A}^3_{\mathbb{C}}$ be an affine algebraic set defined by the equations $F = G = 0$, that is $W = V(F, G)$. One can check that W is irreducible and $\dim W = 1$ (you can use this without proof).
Find the set W^{sing} of singular points on W.

2) Let X and Y be two different irreducible hypersurfaces in $\mathbb{A}^n$. Assume that both X and Y are smooth.
Show that $(X \cup Y)^{sing} = X \cap Y$.

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